Isotropy Generated by Initial Conditions of a System

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Abstract

This article presents a mathematical and conceptual formulation of the hypothesis that the isotropy observed in large-scale physical systems — including the primordial Universe — may emerge directly from extremely restricted initial conditions, characterized by low entropy and probability concentrated in only a few plausible microstates. We examine how such a hypothesis can, in principle, reproduce isotropy without resorting to cosmological inflation, provided that certain conditions on the initial phase-space configuration are satisfied. We also include an analysis of particle density, emergent homogeneity, connections with infinitesimal decompositions of geometric structures, and a formal treatment of the conceptual equation Probability/Entropy = Isotropy, discussing its physically appropriate interpretation.

Keywords: isotropy, low entropy, initial conditions, cosmological inflation, phase space, homogeneity, Banach–Tarski.

1 Introduction

The central objective of this work is to demonstrate mathematically that an early Universe with very low entropy and probability concentrated in a small set of possible microstates can, in principle, exhibit high isotropy without the need for an inflationary process.

This hypothesis depends strongly on *which microstates were allowed* in the initial phase space. That is, it depends on the measure and the physical constraints imposed on the initial set of possibilities. Here, we explicitly present these hypotheses and their dynamical consequences.

2 Phase Space, Entropy, and Initial Probability

Let Γ be the complete phase space of the system (or the primordial Universe). This space contains all possible configurations of the system, that is, all generalized values of position and momentum that characterize each accessible microstate. Each point $x \in \Gamma$ represents a specific microstate (for instance, the position and momentum of all particles).

The initial probability distribution is given by

$$p(x, t_0), \qquad \int_\Gamma p(x, t_0)\, dx = 1. \tag{1}$$

where:

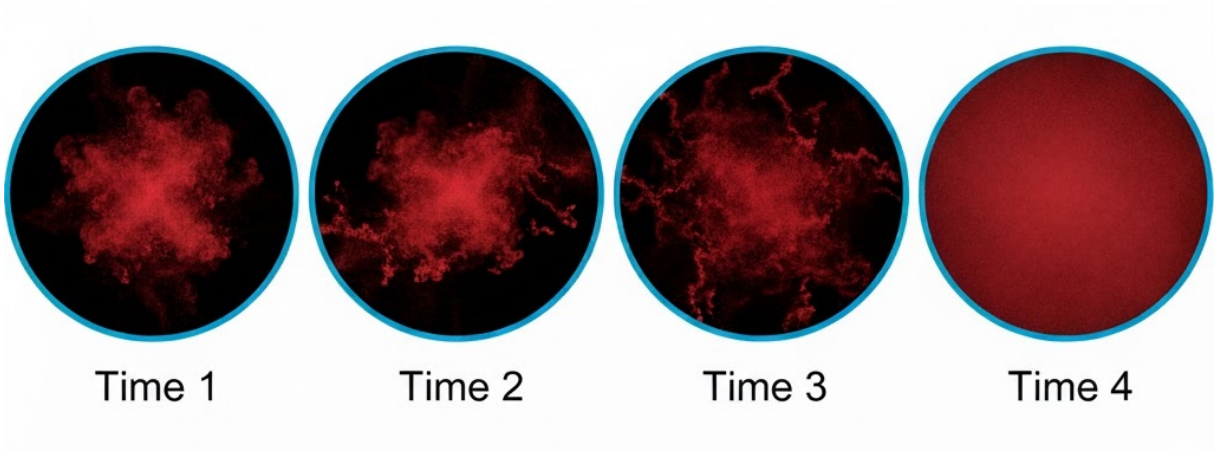

Figure 1: Evolution of particle distribution inside a bounded space

- $p(x, t_0)$ is the probability that the system is in microstate x at the initial instant t_0;
- the integral $\int_\Gamma p(x, t_0)\, dx = 1$ ensures that the distribution is normalized, meaning that the sum of the probabilities of all possible microstates equals 1;
- dx is the infinitesimal volume element in phase space, containing all generalized coordinates.

The entropy associated with the initial distribution is given by the Gibbs entropy:

$$S[p] = -k_B \int_\Gamma p(x) \ln p(x)\, dx. \tag{2}$$

where:

- $S[p]$ is the statistical entropy of the system, measured in joules per kelvin;
- k_B is Boltzmann's constant, which converts between information (logarithms) and physical units of entropy;
- the integrand $p(x) \ln p(x)$ measures the contribution of each microstate to the total statistical disorder;
- the negative sign ensures that more spread-out distributions (greater uncertainty) result in higher entropy.

Low entropy means that the system occupies only a small region of phase space. This can be expressed formally as:

$$\operatorname{supp} p(x, t_0) \subseteq \Gamma_{\text{restricted}}, \qquad \Omega_{\text{effective}} = |\Gamma_{\text{restricted}}| \ll |\Gamma|. \tag{3}$$

where:

- $\operatorname{supp} p(x, t_0)$ ("support of the distribution") is the set of microstates x for which $p(x, t_0) > 0$;
- $\Gamma_{\text{restricted}}$ is a region significantly smaller than the total phase space Γ;

- the notation $|\Gamma_{\text{restricted}}|$ represents the "size" (Liouville volume) of this region in phase space;
- the condition $|\Gamma_{\text{restricted}}| \ll |\Gamma|$ means that only an extremely small fraction of the possible microstates is effectively accessed at the initial instant;
- $\Omega_{\text{effective}}$ is the effective number of relevant microstates: the smaller this number, the lower the entropy of the system.

Thus, a low initial entropy corresponds to a probability highly concentrated in a small number of microstates, characterizing a highly ordered state with preserved directional information.

3 The Conceptual Equation: Probability/Entropy = Isotropy (detailed description)

We propose, as a physical heuristic, the conceptual expression

$$\boxed{\frac{\text{Probability}}{\text{Entropy}} \sim \text{Isotropy}}.$$

Below we provide a precise and operational explanation of each term of this expression, presenting alternative mathematical definitions that allow it to be treated quantitatively.

1. "Probability" (measure of spatial uniformity)

By "Probability" we mean the *uniformity of the spatial distribution* of particles. Let $P(\mathbf{r})$ be the normalized spatial probability density in a volume V:

$$P(\mathbf{r}) \geq 0, \qquad \int_V P(\mathbf{r})\, d^3\mathbf{r} = 1.$$

Two useful ways of quantifying how close $P(\mathbf{r})$ is to the uniform distribution $P_0(\mathbf{r}) = 1/V$:

- **Spatial entropy (Shannon):**

  $$H_{\text{sp}} = -\int_V P(\mathbf{r})\, \ln P(\mathbf{r})\, d^3\mathbf{r},$$

  with maximum $H_{\text{sp}}^{\max} = \ln V$ (native units without k_B). A *normalized uniformity index*:

  $$U \equiv \frac{H_{\text{sp}}}{H_{\text{sp}}^{\max}} \in [0, 1],$$

  where $U \approx 1$ indicates strong spatial uniformity.

- **Kullback–Leibler divergence (alternative):**

  $$D_{\text{KL}}(P\|P_0) = \int_V P(\mathbf{r}) \ln \frac{P(\mathbf{r})}{P_0(\mathbf{r})}\, d^3\mathbf{r},$$

  and we define $U' \equiv e^{-D_{\text{KL}}(P\|P_0)} \in (0, 1]$. The larger U', the more uniform P is.

Both measures capture the intended idea: *the position of a particle becomes irrelevant to the macrostate* when U (or U') is close to 1.

2. "Entropy" (degree of micro/macro disorder)

By "Entropy" we mean the total entropy relevant to the problem — typically the Gibbs entropy of the ensemble over Γ or a sum of partial entropies (spatial, kinetic, gravitational):

$$S_{\text{tot}} = -k_B \int_\Gamma p(x) \ln p(x)\, dx.$$

To make dimensionless quantities comparable, we define a normalized entropy

$$\Sigma \equiv \frac{S_{\text{tot}}}{S_{\max}},$$

where $S_{\max}$ is a chosen reference (for example $k_B \ln \Omega_{\max}$ or another appropriate physical scale). $\Sigma \in [0, 1]$ describes how close the system is to the maximum number of microstates allowed.

3. "Isotropy" (angular symmetry index)

Isotropy is the measure of absence of preferred directions. Let $P(\hat{n})$ be the angular distribution (normalized over the unit sphere $\mathbb{S}^2$) for a relevant observable (e.g., flow directions, orientation of gradients):

$$\int_{\mathbb{S}^2} P(\hat{n})\, d\Omega = 1.$$

We define two indices:

- **Angular entropy:**

$$S_{\text{ang}} = -\int_{\mathbb{S}^2} P(\hat{n}) \ln P(\hat{n})\, d\Omega, \qquad S_{\text{ang}}^{\max} = \ln(4\pi).$$

- **Isotropy index:**

$$I \equiv \frac{S_{\text{ang}}}{S_{\text{ang}}^{\max}} \in [0, 1].$$

  $I \approx 1$ corresponds to nearly perfect angular isotropy ($P(\hat{n}) \approx 1/4\pi$).

4. Operational formulation of the relation $\frac{\textbf{Probability}}{\textbf{Entropy}} \sim$ Isotropy

With the normalized quantities above, we may propose a quantitative and well-defined version of the original heuristic:

$$\boxed{I \;\approx\; \frac{U}{\Sigma} \;=\; \frac{H_{\text{sp}}/H_{\text{sp}}^{\max}}{S_{\text{tot}}/S_{\max}}}.$$

Interpretation:

- Numerator U (spatial uniformity) measures how irrelevant particle position becomes for the macrostate.
- Denominator Σ measures the relative degree of exploration of phase space (normalized total entropy).

- When U is high (almost uniform spatial distribution) and Σ is low (initial ensemble restricted to homogeneous microstates), I becomes large, indicating high isotropy — this is the scenario of isotropy emerging from *initial conditions*.

- Alternatively, in late regimes where S_{tot} approaches $S_{\max}$ (high global entropy), angular isotropy may also emerge because $P(\hat{n})$ tends to uniformize; in this case the effective dependence is better expressed as

$$I \sim U \times F(\Sigma),$$

with $F(\Sigma)$ a function increasing with Σ (e.g. $F(\Sigma) \approx \Sigma$ or $F(\Sigma) \approx 1$, depending on dynamics).

5. Notes on units, scales, and limitations

1. All quantities U, Σ, I were defined dimensionless and normalized to allow comparisons.

2. The form $I \approx U/\Sigma$ is a *model proposition* rather than a universal law — its utility is heuristic: it explains how high spatial uniformity combined with low effective entropy leads to initial isotropy, while regimes of maximal entropy yield isotropy for different statistical reasons.

3. Dynamical dependence: U and Σ are functions of time t. The evolution of $I(t)$ depends on stability of the support of $p(x,t)$ and on amplification/attenuation of anisotropic modes by the dynamics (see evolution section).

4. Alternative metrics (total variation, KL divergence, harmonic spectrum a_{lm}) should be tested numerically to calibrate the relation.

6. Synthesis

The original expression $\frac{\text{Probability}}{\text{Entropy}} \sim$ Isotropy becomes operational and measurable when we define:

$$\text{Probability} \mapsto U \equiv \frac{H_{\text{sp}}}{H_{\text{sp}}^{\max}}, \qquad \text{Entropy} \mapsto \Sigma \equiv \frac{S_{\text{tot}}}{S_{\max}}, \qquad \text{Isotropy} \mapsto I \equiv \frac{S_{\text{ang}}}{S_{\text{ang}}^{\max}}.$$

With these definitions, the heuristic becomes the model relation

$$I \approx \frac{U}{\Sigma},$$

subject to the observations and remarks above. This formulation allows numerical testing and theoretical analysis of scenarios in which isotropy may emerge from initial conditions (low entropy and high spatial uniformity) or, alternatively, from statistical relaxation to maximal entropy.

- **Probability**: the particle may be located anywhere without altering macroscopic homogeneity. At maximal entropy, the particle's individual position does not affect the macrostate.

- **Entropy**: degree of microscopic disorder and number of microstates compatible with the macrostate. A space highly chaotic at both micro and macro scales.
- **Isotropy**: absence of preferred directions.

This relation is formalized through the uniform spatial distribution

$$P(\mathbf{r}) = \frac{1}{V}$$

and the maximization of entropy under global constraints. It expresses that the combination of a highly uniform spatial distribution and a restricted set of accessible microstates leads to an isotropic state, in which no preferred directions exist and the system exhibits emergent angular symmetry. Entropy, in this context, functions as a measure of the dispersion of microscopic possibilities, whereas probability describes the uniformity with which such possibilities are distributed across the considered volume.

The treatment presented shows that, under strongly organized initial conditions, isotropy arises as a direct consequence of the statistical structure of the system, reinforcing the interdependence between low entropy, probabilistic homogeneity, and the absence of anisotropies. This approach highlights the capacity of ordered initial conditions to generate, on their own, widely isotropic configurations, consistent with the symmetries expected in extensive physical systems.

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